

Eisenstein series attached to small automorphic representations

Henrik Gustafsson

Representation Theory Seminar
Stanford 2016

 hgustafsson.se

Based on

Small automorphic representations and degenerate Whittaker vectors

HG, Axel Kleinschmidt, Daniel Persson

[arXiv:1412.5625](https://arxiv.org/abs/1412.5625) [math.NT]

[GKP14]

Journal of Number Theory 166 (Sep, 2016) 344–399

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$SL(n)$

E_6, E_7, E_8

Outline

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- Why do string theorists study Eisenstein series?

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- How can we compute them?
- What happens for *small automorphic representations*?

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- Why do *string theorists* study Eisenstein series?
- What kind of *Fourier coefficients* are we interested in?
- How can we compute them?
- What happens for *small automorphic representations*?
- What's next?

String theory



String theory



String theory



String theory



World-sheet
 Σ

String theory



World-sheet
 Σ

Typical string length: l_s



String theory



World-sheet
 Σ

Typical string length: l_s



$$\alpha' = l_s^2$$

String theory

Space-time is described by a Riemannian manifold M

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String theory = dynamics of the embedding maps

$$X : \Sigma \rightarrow M$$

world-sheet space-time

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We will focus
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Consistency requires: 10-dimensional M

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Space-time is described by a Riemannian manifold M

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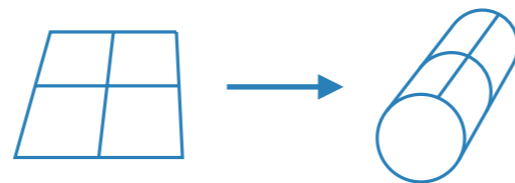
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world-sheet space-time

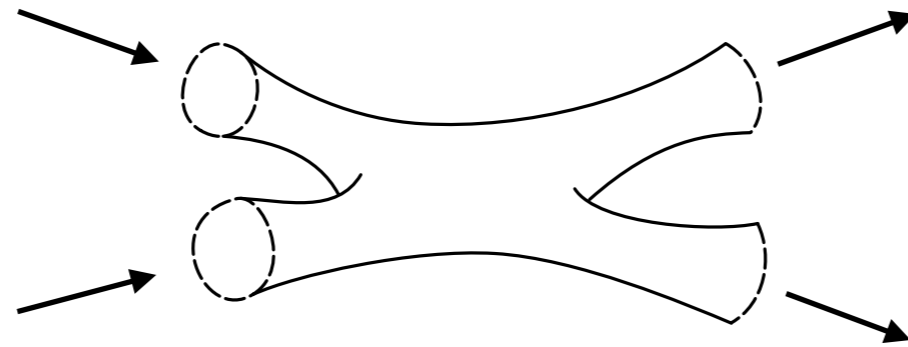
Consistency requires: 10-dimensional M

Toroidal compactifications

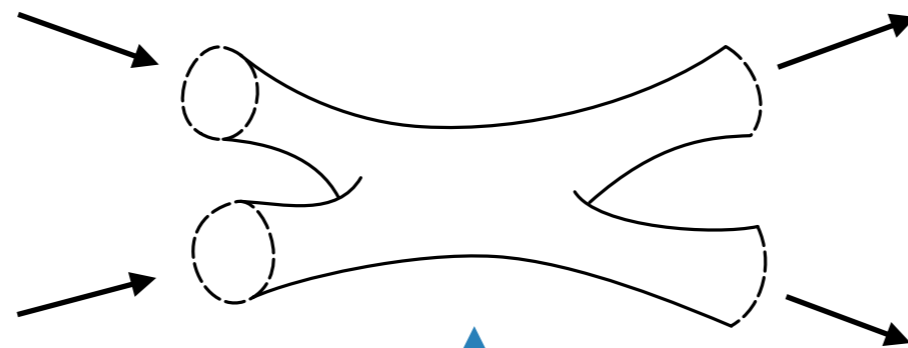


D dimensions

Interactions

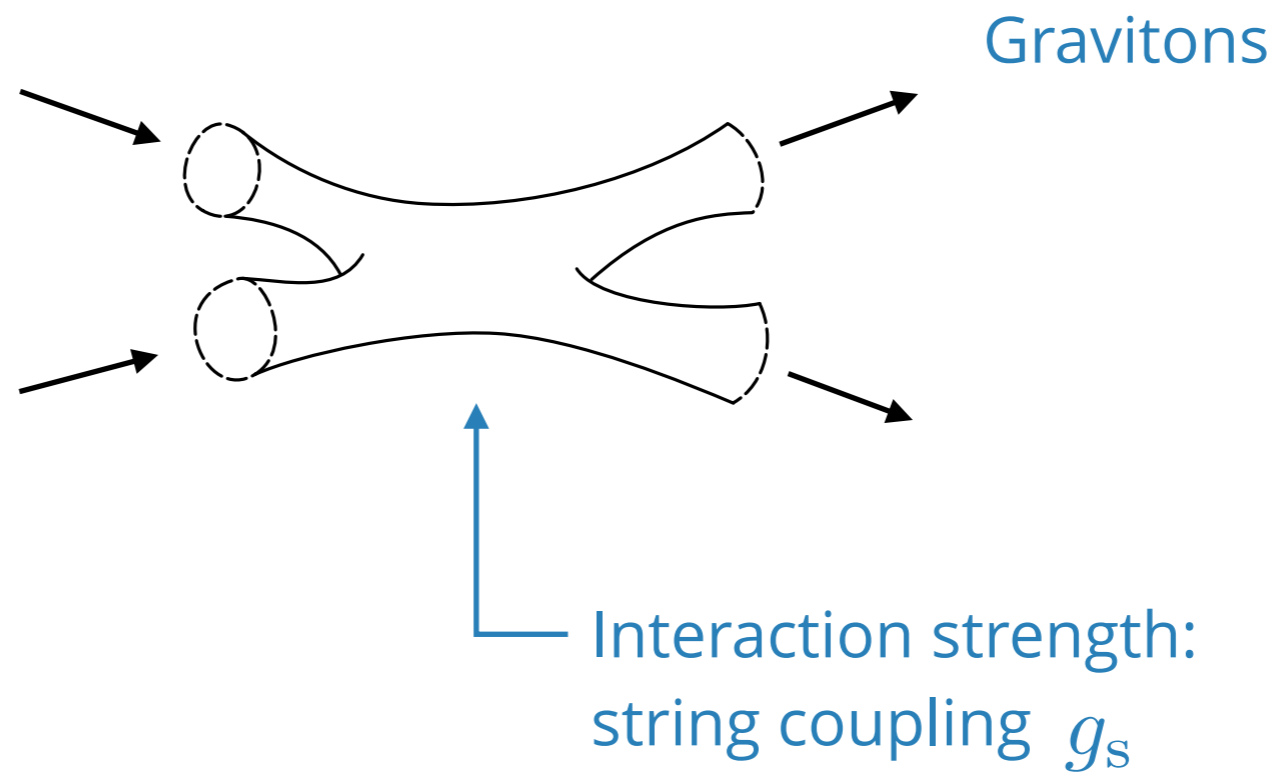


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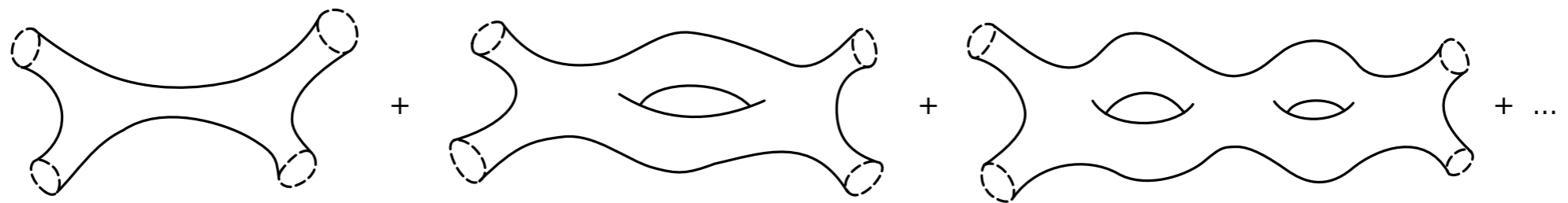


Interaction strength:
string coupling g_s

Interactions

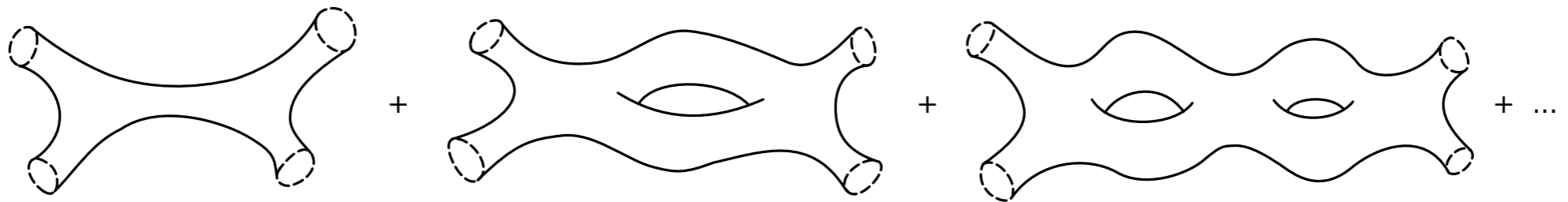


Interactions



Interactions

Weighted by: $g_s^{-\chi_E}$ $-\chi_E = 2(\text{genus} - 1) + \text{boundaries}$

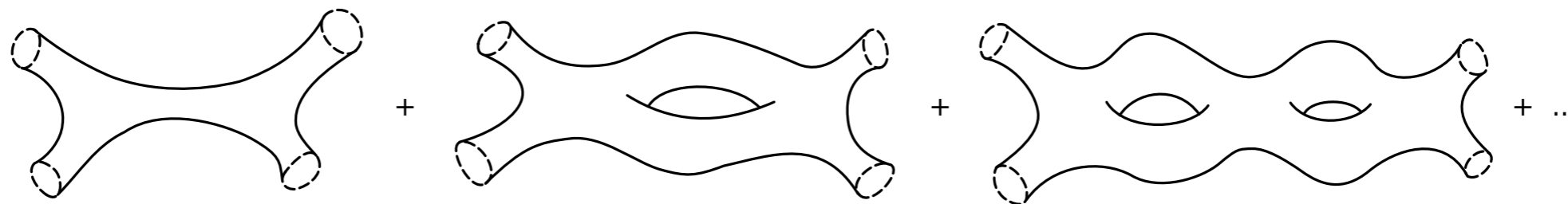


Interactions

Weighted by: $g_s^{-\chi_E}$

Euler characteristic

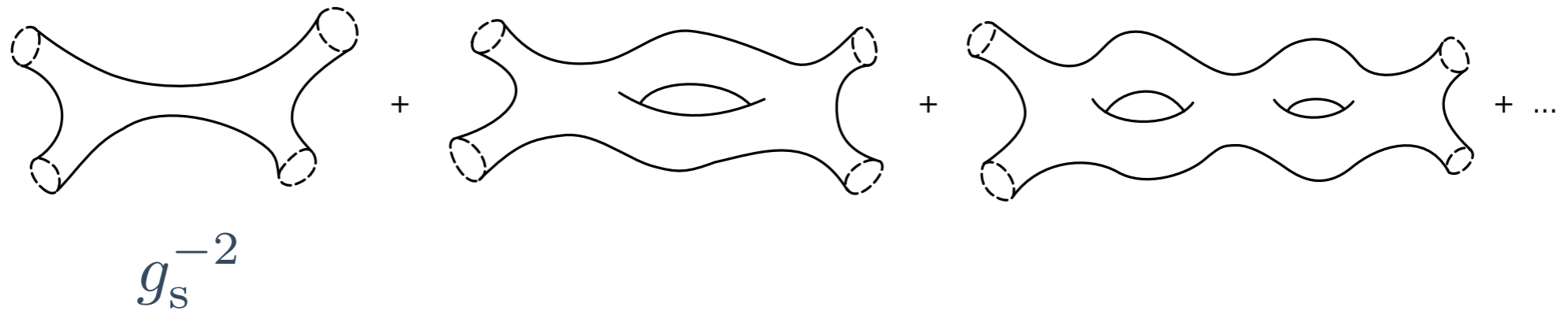
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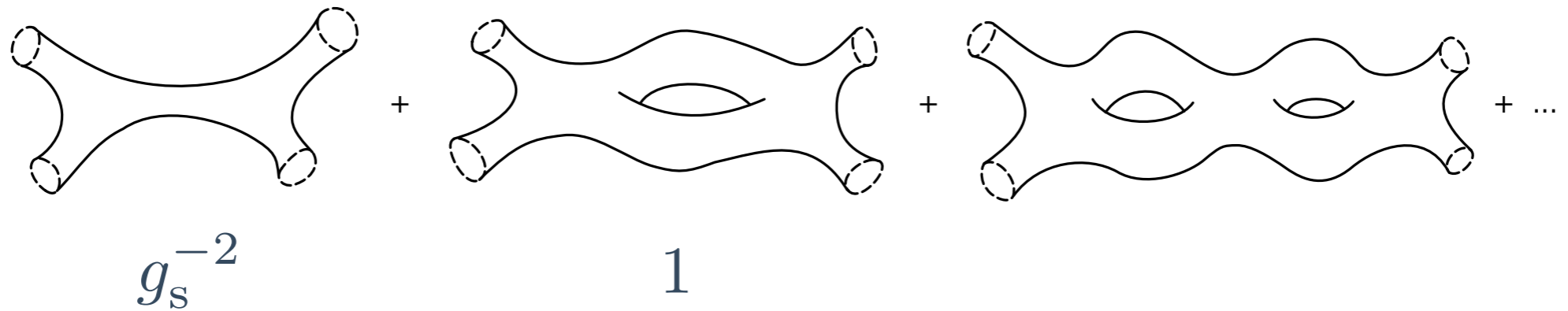


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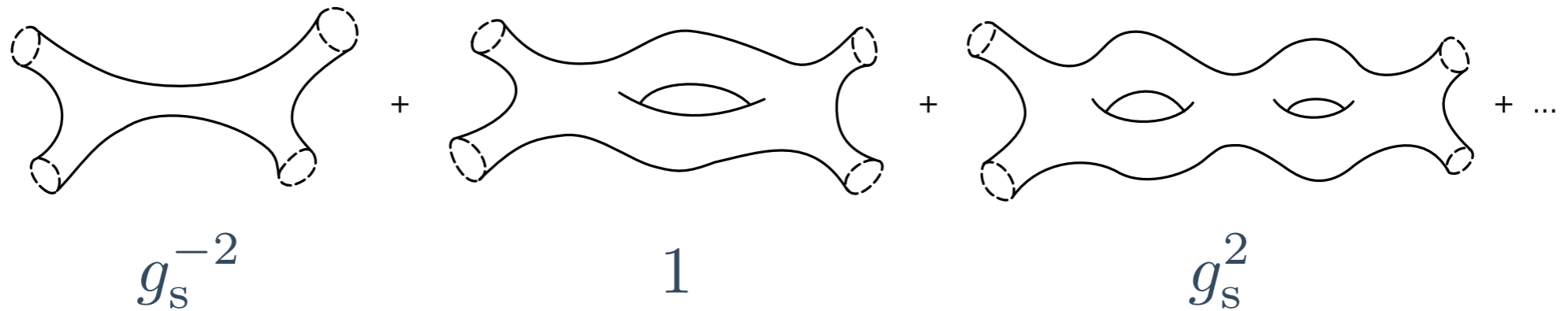


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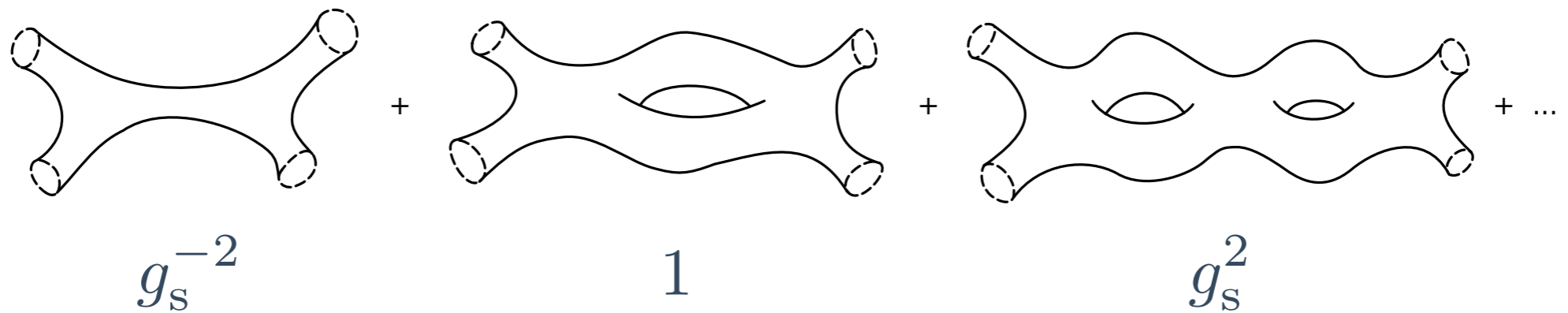
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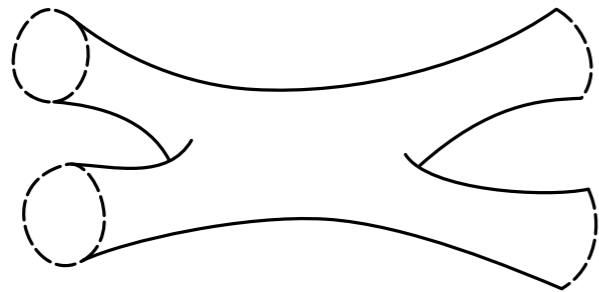
Euler characteristic
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Perturbative expansion in g_s

Interactions

Gravitons in D dimensions



Interactions

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Interactions

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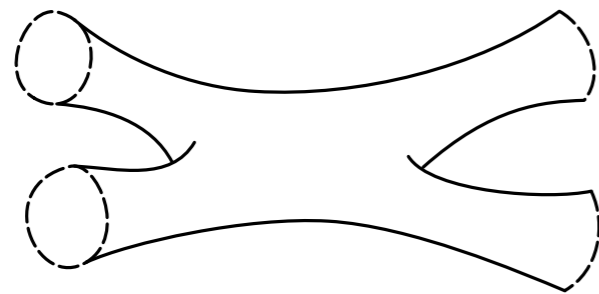


$$R + (\alpha')^3 \mathcal{E}_0^{(D)}(g) R^4 + (\alpha')^5 \mathcal{E}_4^{(D)}(g) D^4 R^4 + (\alpha')^6 \mathcal{E}_6^{(D)}(g) D^6 R^4 + \dots$$

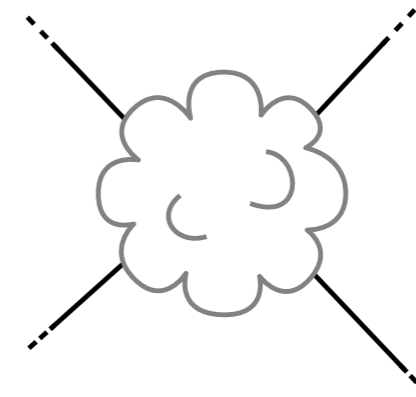
↑
Expansion
parameter

Interactions

Gravitons in D dimensions



Effective field theory



Einstein gravity

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Contractions of derivatives and 4 Riemann tensors (known)

Interactions

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Moduli space

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D	$G(\mathbb{R})$	K
10	$SL(2, \mathbb{R})$	$SO(2)$
9	$SL(2, \mathbb{R}) \times \mathbb{R}^+$	$SO(2)$
8	$SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$	$SO(3) \times SO(2)$
7	$SL(5, \mathbb{R})$	$SO(5)$
6	$Spin(5, 5; \mathbb{R})$	$(Spin(5) \times Spin(5))/\mathbb{Z}_2$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$
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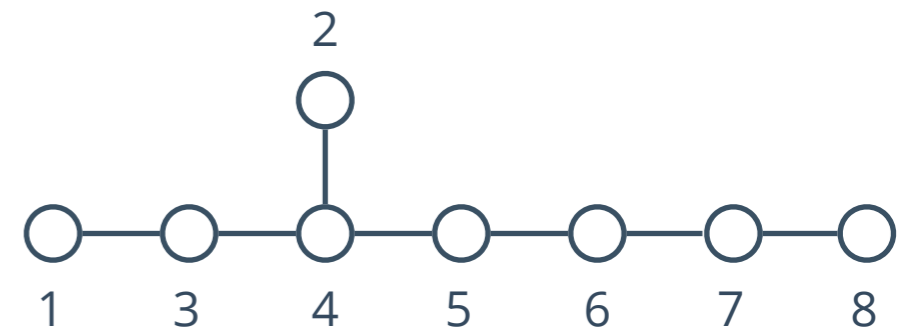
[Cremmer-Julia]

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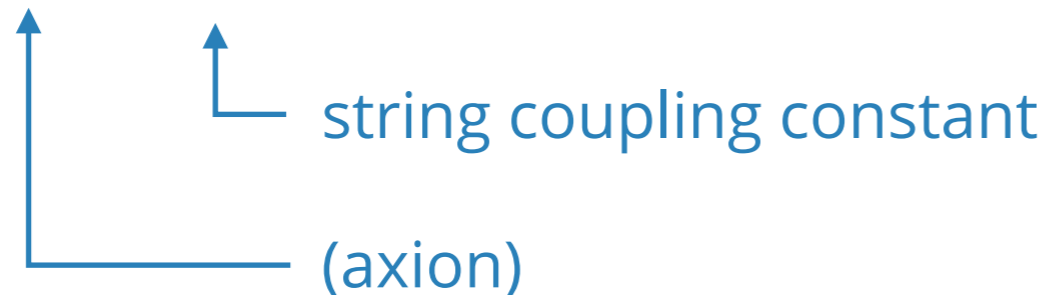
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 string coupling constant

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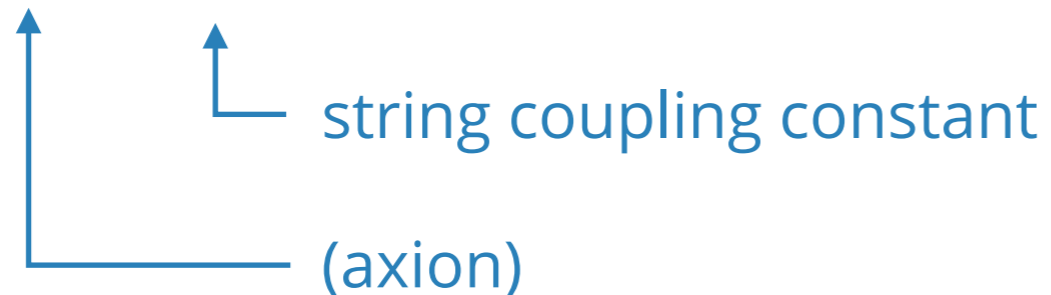
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$$\mathcal{E}_n(\tau) = \mathcal{E}_n^{(10)}(g)$$

U-duality

$G(\mathbb{R}) \curvearrowright \mathcal{M}_{\text{classical}}$ classical symmetry

[Hull-Townsend]

U-duality

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Quantization of charges

[Hull-Townsend]

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Quantization of charges \implies classical symmetry \longrightarrow discrete symmetry

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Chevalley group $G(\mathbb{Z})$

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[Hull-Townsend]

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All observables are invariant under $G(\mathbb{Z})$

[Hull-Townsend]

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$$\mathcal{E}_0^{(D)}(g), \mathcal{E}_4^{(D)}(g), \mathcal{E}_6^{(D)}(g) : G(\mathbb{Z}) \backslash G(\mathbb{R}) / K \rightarrow \mathbb{C}$$

Automorphic forms

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- (B) φ is an eigenfunction under right-translations of $k \in K$

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- (C) φ is an eigenfunction to all G -invariant differential operators

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- (B) K-finiteness: $\dim(\text{span}\{\varphi(gk) \mid k \in K\}) < \infty$
- (C) Z-finiteness: $\dim(\text{span}\{X\varphi(g) \mid X \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})\}) < \infty$

$\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ is the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$

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- (C) Z-finiteness: $\dim(\text{span}\{X\varphi(g) \mid X \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})\}) < \infty$
- (D) Growth: for any norm $\|\cdot\|$ on $G(\mathbb{R})$ there exists a positive integer n and constant C such that $|\varphi(g)| \leq C\|g\|^n$

$\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ is the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$

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- (A) Automorphic invariance: ✓ U-duality
- (B) K-finiteness:
- (C) Z-finiteness:
- (D) Growth:

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- (B) K-finiteness: ✓ spherical
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- (D) Growth: ✓ weak coupling limit from string perturbation theory

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- (C) Z-finiteness: ?
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Supersymmetry constraints



Supersymmetry constraints



10 dimensions: $\Delta = 4\tau_2^2 \frac{\partial}{\partial\tau} \frac{\partial}{\partial\bar{\tau}}$ Laplacian

Supersymmetry constraints



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$$\left(\Delta - \frac{3}{4}\right) \mathcal{E}_0(\tau) = 0$$

[Green-Sethi]

Supersymmetry constraints



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[Green-Sethi]

$$(\Delta - \frac{15}{4})\mathcal{E}_4(\tau) = 0$$

[Sinha]

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$$(\Delta - \frac{3}{4})\mathcal{E}_0(\tau) = 0$$

[Green-Sethi]

$$(\Delta - \frac{15}{4})\mathcal{E}_4(\tau) = 0$$

[Sinha]

$$(\Delta - 12)\mathcal{E}_6(\tau) = -(\mathcal{E}_0(\tau))^2$$

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Not an automorphic form in a strict sense

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Not an automorphic form in a strict sense

Similarly for lower dimensions

Eisenstein series

$$E(s; \tau) =$$

$$s \in \mathbb{C}$$

Eisenstein series

$$E(s; \tau) = \sum_{\substack{c, d \in \mathbb{Z} \\ (c, d) \neq (0, 0)}} \frac{\text{Im}(\tau)^s}{|c\tau + d|^{2s}}$$

$$s \in \mathbb{C}$$

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$$E(s; \tau) = \frac{1}{2\zeta(2s)} \sum_{\substack{c, d \in \mathbb{Z} \\ (c, d) \neq (0, 0)}} \frac{\text{Im}(\tau)^s}{|c\tau + d|^{2s}}$$

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$$B(\mathbb{Z}) = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in SL(2, \mathbb{Z}) \right\} \quad \text{Borel subgroup}$$

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Completed Riemann zeta function

$$\xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$$

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Bessel function
of the second kind

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Eisenstein series

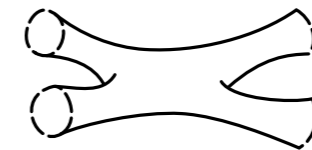
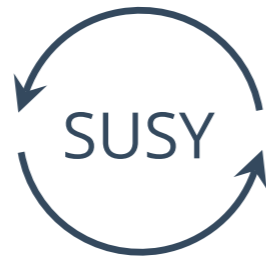
$$(\Delta - s(s - 1))E(s; \tau) = 0 \quad E(s; \tau) \sim \tau_2^s \quad g_s = \tau_2^{-1} \rightarrow 0$$

[Green-Gutperle, Pioline, Green-Russo-Vanhove]

Eisenstein series

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$$\mathcal{E}_0(\tau) \sim 2\zeta(3)\tau_2^{3/2}$$

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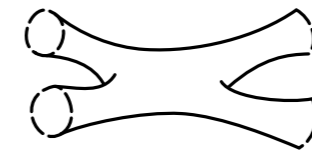
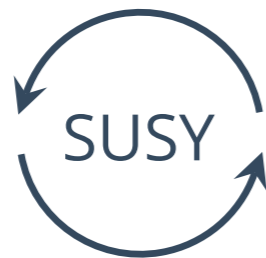
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[Green-Gutperle, Pioline, Green-Russo-Vanhove]

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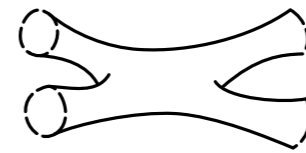
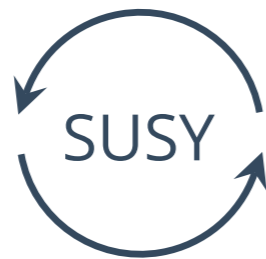
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[Green-Gutperle, Pioline, Green-Russo-Vanhove]

Eisenstein series



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$\mathcal{E}_6(\tau)$ as a sum over images $\sum_{B(\mathbb{Q}) \backslash G(\mathbb{Z})}$ but not of a character χ

[Green-Miller-Vanhove]

Extracting physical information

Expand Bessel function in g_s

$$\tau = \chi + ig_s^{-1}$$

Extracting physical information

Expand Bessel function in g_s $\tau = \chi + ig_s^{-1}$

 Interaction strength

Extracting physical information

Expand Bessel function in g_s $\tau = \chi + ig_s^{-1}$

The diagram illustrates the expansion of a Bessel function in terms of the interaction strength g_s . The equation $\tau = \chi + ig_s^{-1}$ is shown, where χ is identified as the axion. Blue arrows point from the text labels to the corresponding terms in the equation: one arrow points from "Interaction strength" to g_s , and another points from "(axion)" to χ .

Interaction strength

(axion)

Extracting physical information

Expand Bessel function in g_s $\tau = \chi + ig_s^{-1}$
Interaction strength (axion)

$$\mathcal{E}_0(\tau) = 2\zeta(3)g_s^{-3/2} + 4\zeta(2)g_s^{1/2} + 2\pi \sum_{m \neq 0} \sqrt{|m|} \sigma_{-2}(m) e^{-2\pi|m|g_s^{-1} + 2\pi im\chi} \left[1 + \mathcal{O}(g_s) \right]$$

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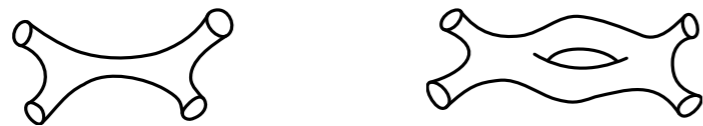
.....

Perturbative
(zero-mode)

[Green-Gutperle]

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Perturbative
(zero-mode)

Non-perturbative
(remaining modes)

$$e^{-\frac{1}{g_s}}$$

[Green-Gutperle]

Extracting physical information

Expand Bessel function in g_s $\tau = \chi + ig_s^{-1}$
↑ ↑
Interaction strength (axion)



Instanton action

$$\mathcal{E}_0(\tau) = 2\zeta(3)g_s^{-3/2} + 4\zeta(2)g_s^{1/2} + 2\pi \sum_{m \neq 0} \sqrt{|m|} \sigma_{-2}(m) e^{-2\pi|m|g_s^{-1} + 2\pi im\chi} \left[1 + \mathcal{O}(g_s) \right]$$

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Sums over the number of ways the charge m can be factorised into two integers

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Sums over the number of ways the charge m can be factorised into two integers

wrapping number and charge
of a T-dual D-particle

[Green-Gutperle]

Lower dimensions

Lower dimensions

D	$G(\mathbb{R})$	K	$G(\mathbb{Z})$
10	$SL(2, \mathbb{R})$	$SO(2)$	$SL(2, \mathbb{Z})$
9	$SL(2, \mathbb{R}) \times \mathbb{R}^+$	$SO(2)$	$SL(2, \mathbb{Z}) \times \mathbb{Z}_2$
8	$SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$	$SO(3) \times SO(2)$	$SL(3, \mathbb{Z}) \times SL(2, \mathbb{Z})$
7	$SL(5, \mathbb{R})$	$SO(5)$	$SL(5, \mathbb{Z})$
6	$Spin(5, 5; \mathbb{R})$	$(Spin(5) \times Spin(5)) / \mathbb{Z}_2$	$Spin(5, 5; \mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8) / \mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU(8) / \mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$Spin(16) / \mathbb{Z}_2$	$E_8(\mathbb{Z})$

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Parabolic subgroups

Fourier expand
in different directions



Unipotent subgroup U

Parabolic subgroups

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Unipotent subgroup U



Choice of parabolic subgroup P

Parabolic subgroups

Fourier expand
in different directions



Unipotent subgroup U



Choice of parabolic subgroup P

Σ choice of simple roots

$\langle \Sigma \rangle$ generated root system

Parabolic subgroups

Fourier expand
in different directions



Unipotent subgroup U



Choice of parabolic subgroup P

Σ choice of simple roots

$\langle \Sigma \rangle$ generated root system

$$\mathfrak{g}_\alpha = \{g \in \mathfrak{g} \mid [h, g] = \alpha(h)g \quad \forall h \in \mathfrak{h}\}$$

Parabolic subgroups

Fourier expand
in different directions



Unipotent subgroup U



Choice of parabolic subgroup P

Σ choice of simple roots

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Cartan subalgebra

Parabolic subgroups

Fourier expand
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Cartan subalgebra

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{p})} \mathfrak{g}_\alpha$$

Parabolic subgroups

Fourier expand
in different directions



Unipotent subgroup U



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Parabolic subgroups

Fourier expand
in different directions



Unipotent subgroup U



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Positive roots

Parabolic subgroups

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Levi decomposition

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Levi decomposition

$$\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$$

Parabolic subgroups

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Levi decomposition

$$\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u} \quad \mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \langle \Sigma \rangle} \mathfrak{g}_{\alpha} \quad \mathfrak{u} = \bigoplus_{\alpha \in \Delta(\mathfrak{u})} \mathfrak{g}_{\alpha}$$

Parabolic subgroups

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{p})} \mathfrak{g}_{\alpha} \quad \Delta(\mathfrak{p}) = \Delta_+ \cup \langle \Sigma \rangle$$

Levi decomposition

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Corresponding group P

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$$P = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} \right\} = LU$$

Parabolic subgroups

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Maximal parabolic

Parabolic subgroups

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Minimal parabolic
Borel



Maximal parabolic

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Corresponding group P



Minimal parabolic
Borel

$$B = NA$$

$$N = \left\{ \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \right\}$$



Maximal parabolic

$$P = LU$$

$$U = \left\{ \begin{pmatrix} 1 & & * \\ & 1 & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \right\}$$

Fourier expansion

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Let $\psi : U(\mathbb{Z}) \backslash U(\mathbb{R}) \rightarrow U(1)$ be a multiplicative character

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$$u = \prod_{\alpha \in \Delta^{(1)}(\mathfrak{u})} \exp(u_\alpha E_\alpha) \mapsto \exp(2\pi i \sum_{\alpha \in \Delta^{(1)}(\mathfrak{u})} m_\alpha u_\alpha) \quad m_\alpha \in \mathbb{Z} \text{ charges}$$

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Multiplicative: $\Delta^{(1)}(\mathfrak{u}) = \Delta(\mathfrak{u}) \backslash \Delta([\mathfrak{u}, \mathfrak{u}])$

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Fourier expansion

Fourier expansion

$$E(\chi; g) = \sum_{\psi} F_U(\chi, \psi; g)$$

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Fourier expansion

$$E(\chi; g) = F_U(\chi, 1; g) + \sum_{\psi^{(1)} \neq 1} F_{U^{(1)}}(\chi, \psi^{(1)}; g) + \sum_{\psi^{(2)} \neq 1} F_{U^{(2)}}(\chi, \psi^{(2)}; g) + \dots$$

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$$U^{(1)} = U \quad U^{(n+1)} = [U^{(n)}, U^{(n)}]$$

Terminology

$P = B \longrightarrow U = N$ Fourier coefficient is a Whittaker coefficient

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F_U

W_N

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F_U

W_N

Characters and coefficients with all $m_\alpha \neq 0$ are called **generic**
otherwise they are called **degenerate**

Fourier expansion

Choice of unipotent subgroup U \longleftrightarrow Study different perturbative and non-perturbative effects

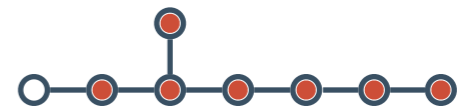
[Green-Miller-Vanhove]

Fourier expansion

Choice of unipotent subgroup U \longleftrightarrow Study different perturbative and non-perturbative effects

- String perturbation limit
D-instantons | NS5-instantons

$$g_s \rightarrow 0$$



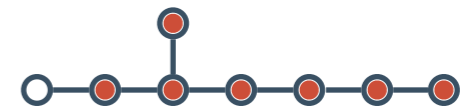
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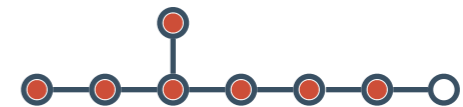
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- Decompactification limit
Higher dimensional black holes | BPS states

Large radius for
compactified circle



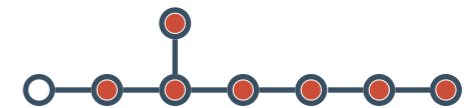
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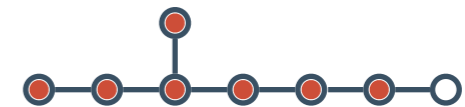
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- M-theory limit
M2, M5-instantons

Large M-theory torus



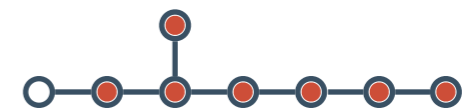
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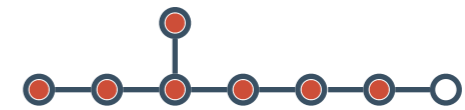
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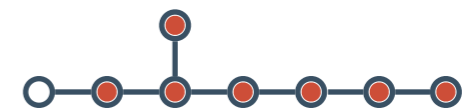
Maximal parabolic
subgroups

Fourier expansion

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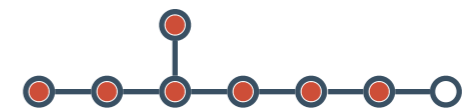
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[Green-Miller-Vanhove]

Maximal parabolic
subgroups

Difficult to compute!

Fourier expansion

Goal: find expressions for Fourier coefficients
in terms of (known) Whittaker coefficients

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Adelic framework

*An **efficient**, but abstract, way to approach the subject of automorphic forms is by the introduction of **adeles**, rather **ungainly objects** that nevertheless, once familiar, **spare** much unnecessary thought and **many useless calculations**.*

— Robert P. Langlands*

*Representation theory - its rise and its role in number theory, Proceedings of the Gibbs symposium (1989)

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The adeles

For a prime p

\mathbb{Q}

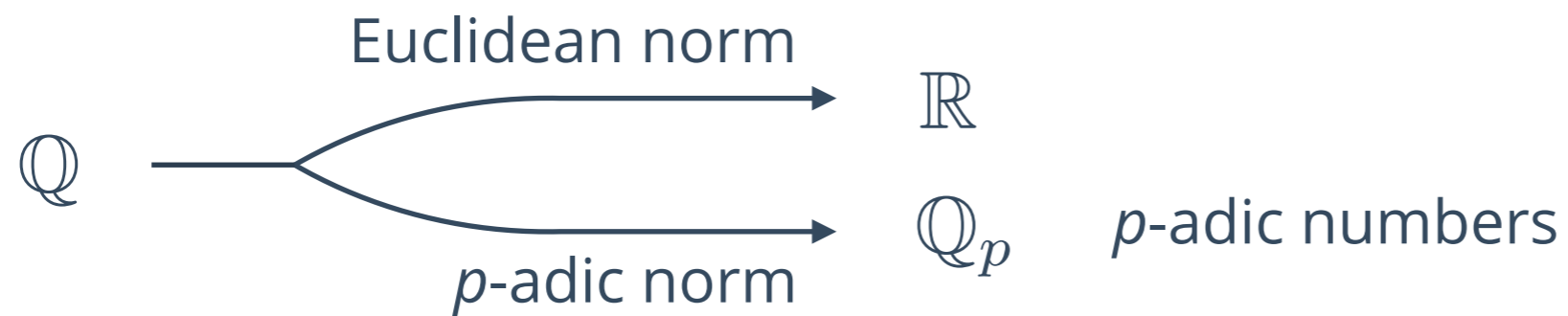
The adeles

For a prime p



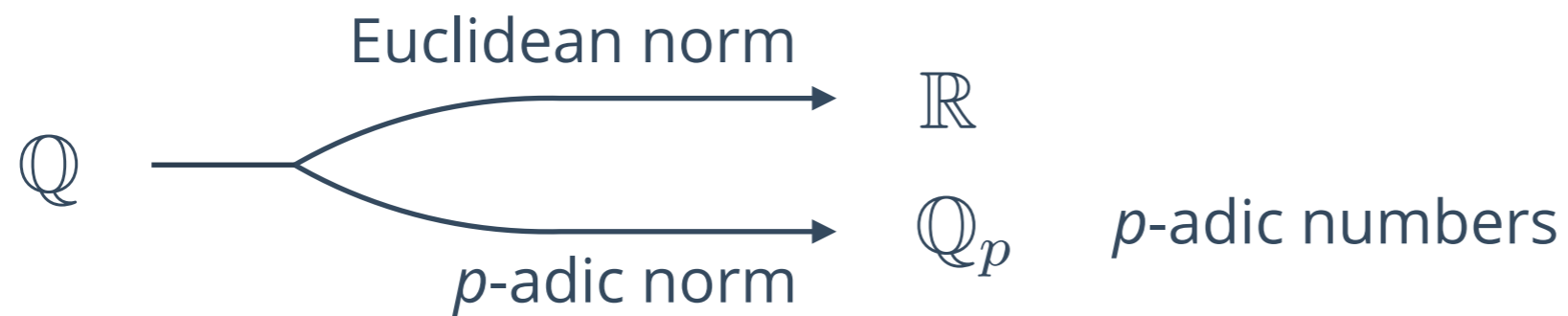
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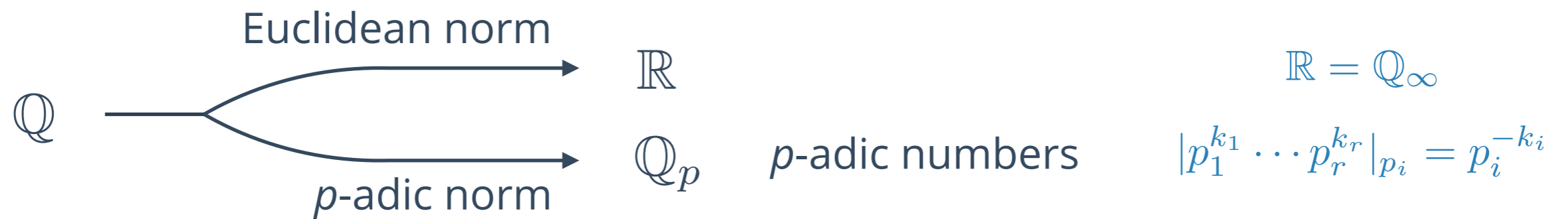
For a prime p



$$\mathbb{R} = \mathbb{Q}_\infty$$

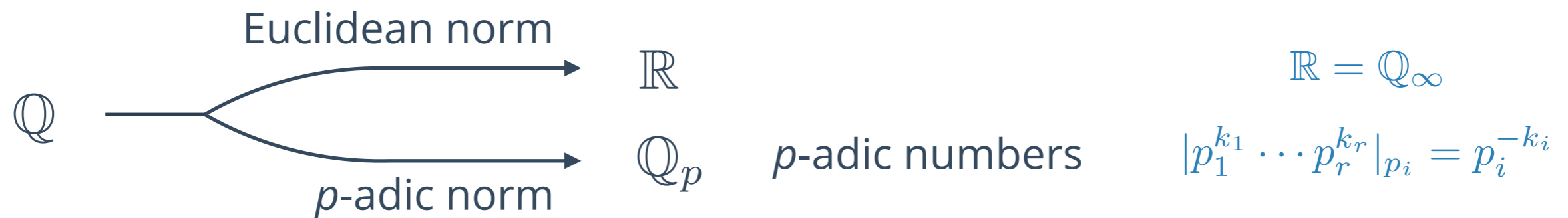
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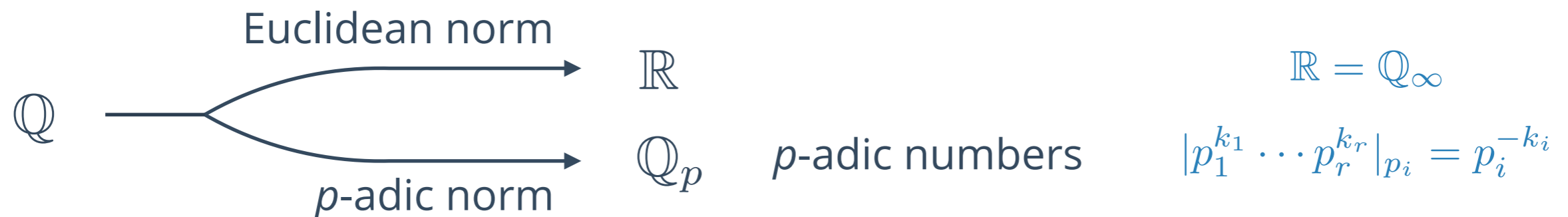


The adeles are then defined as

$$\mathbb{A} = \mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \prod'_{p \text{ prime}} \mathbb{Q}_p \quad x = (x_\infty; x_2, x_3, x_5, \dots) \in \mathbb{A}$$

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$$\mathbb{Q} \hookrightarrow \mathbb{A}$$

$$q \mapsto (q; q, q, \dots)$$

\mathbb{Q} is discrete in \mathbb{A} taking the role of \mathbb{Z} in \mathbb{R}

Much easier to work with since it is a field!

Adelic framework

$$\mathcal{E}_0^{(D)}(g), \mathcal{E}_4^{(D)}(g), \mathcal{E}_6^{(D)}(g) : G(\mathbb{Z}) \backslash G(\mathbb{R}) / K \rightarrow \mathbb{C}$$

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Lift to the adèles

[FGKP15 §4.2.2]

$$G(\mathbb{A}) = G(\mathbb{R}) \times \prod'_{p \text{ prime}} G(\mathbb{Q}_p) \quad K_{\mathbb{A}} = K \times \prod_{p \text{ prime}} G(\mathbb{Z}_p)$$

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Eisenstein series \longrightarrow Adelic Eisenstein series

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$$\sum_{\gamma \in B(\mathbb{Z}) \backslash G(\mathbb{Z})} \chi_{\mathbb{R}}(\gamma g)$$

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Fourier coefficients \longrightarrow Adelic Fourier coefficients

Adelic framework

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Fourier coefficients \longrightarrow Adelic Fourier coefficients

$$\int_{U(\mathbb{Z}) \backslash U(\mathbb{R})} E(\chi; ug) \overline{\psi_{\mathbb{R}}(u)} du$$

$$\int_{U(\mathbb{Q}) \backslash U(\mathbb{A})} E(\chi; ug) \overline{\psi_{\mathbb{A}}(u)} du$$

$$m_{\alpha} \in \mathbb{Z}$$

$$m_{\alpha} \in \mathbb{Q}$$

Computing adelic Fourier coefficients

[FGKP15 §9-10]

Whittaker coefficients

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Whittaker coefficients

Constant term: Langlands' constant term formula

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Computing adelic Fourier coefficients

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Maximally degenerate

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[FGKP15 §9-10]

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Computing adelic Fourier coefficients

[FGKP15 §9-10]

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Factorisation

[GKP14]

Fourier coefficients

In terms of Whittaker coefficients

Simplify drastically for certain χ

Example of simplifications

$$G = SL(3)$$

$$E(\chi; g)$$

$$\chi \longleftrightarrow (s_1, s_2) \in \mathbb{C}^2$$

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$$\begin{matrix} m_1 & m_2 \\ \circ & \text{---} & \circ \end{matrix}$$

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$$W_N(\chi, \psi_{m_1, m_2}; g) \propto \left(\begin{matrix} \text{arithmetic} \\ \text{factor} \end{matrix} \right) \int K_{\#}(\dots) K_{\#}(\dots)$$

[FGKP15 §10.6]

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Vanishes for certain (s_1, s_2)

[FGKP15 §10.6]

Example of simplifications

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Automorphic representations

$G(\mathbb{A}) \curvearrowright$ Space of automorphic forms*

* With some subtleties described in [FGKP15 §6]

[Bump, Goldfeld-Hundley]

Automorphic representations

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Automorphic representation π = an irreducible component of the above space under this action

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What is a small automorphic representation?

* With some subtleties described in [FGKP15 §6]

[Bump, Goldfeld-Hundley]

Wavefront set

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg,
Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

Wavefront set

The (global) wavefront set contains all the characters ψ which can give rise to non-vanishing Fourier coefficients in that representation

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Small automorphic representations have few non-vanishing Fourier coefficients

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

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Characters $\psi \longleftrightarrow$ Nilpotent elements in $\mathfrak{g}(\mathbb{Q})$

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Closure with respect to partial ordering

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Nilpotent orbits

[Collingwood-McGovern]

For $SL(n)$, orbits can be identified
with partitions of n

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(p_1, p_2, \dots)

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 $(p_1, p_2, \dots) \leq (q_1, q_2, \dots)$ partial ordering

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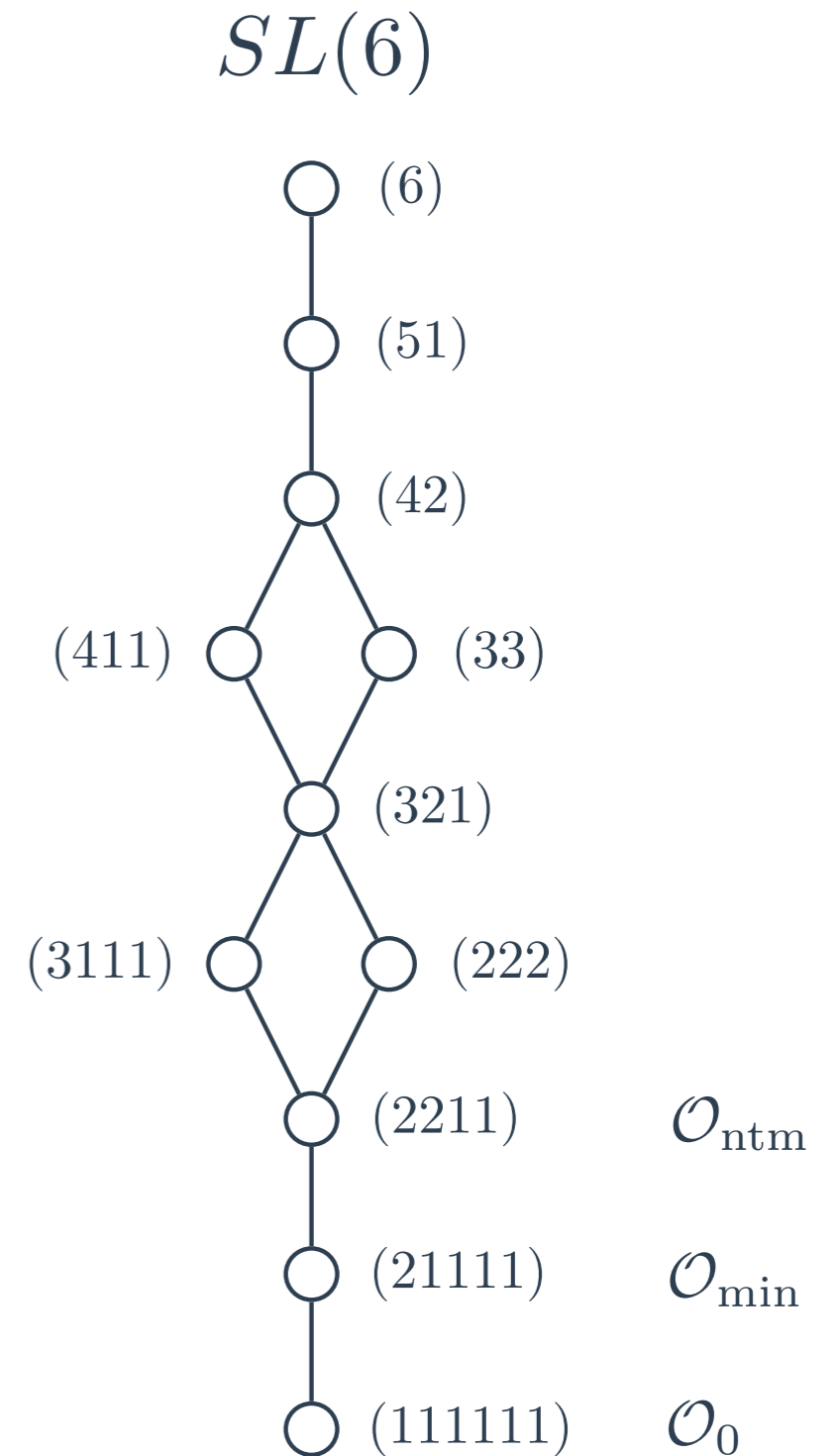
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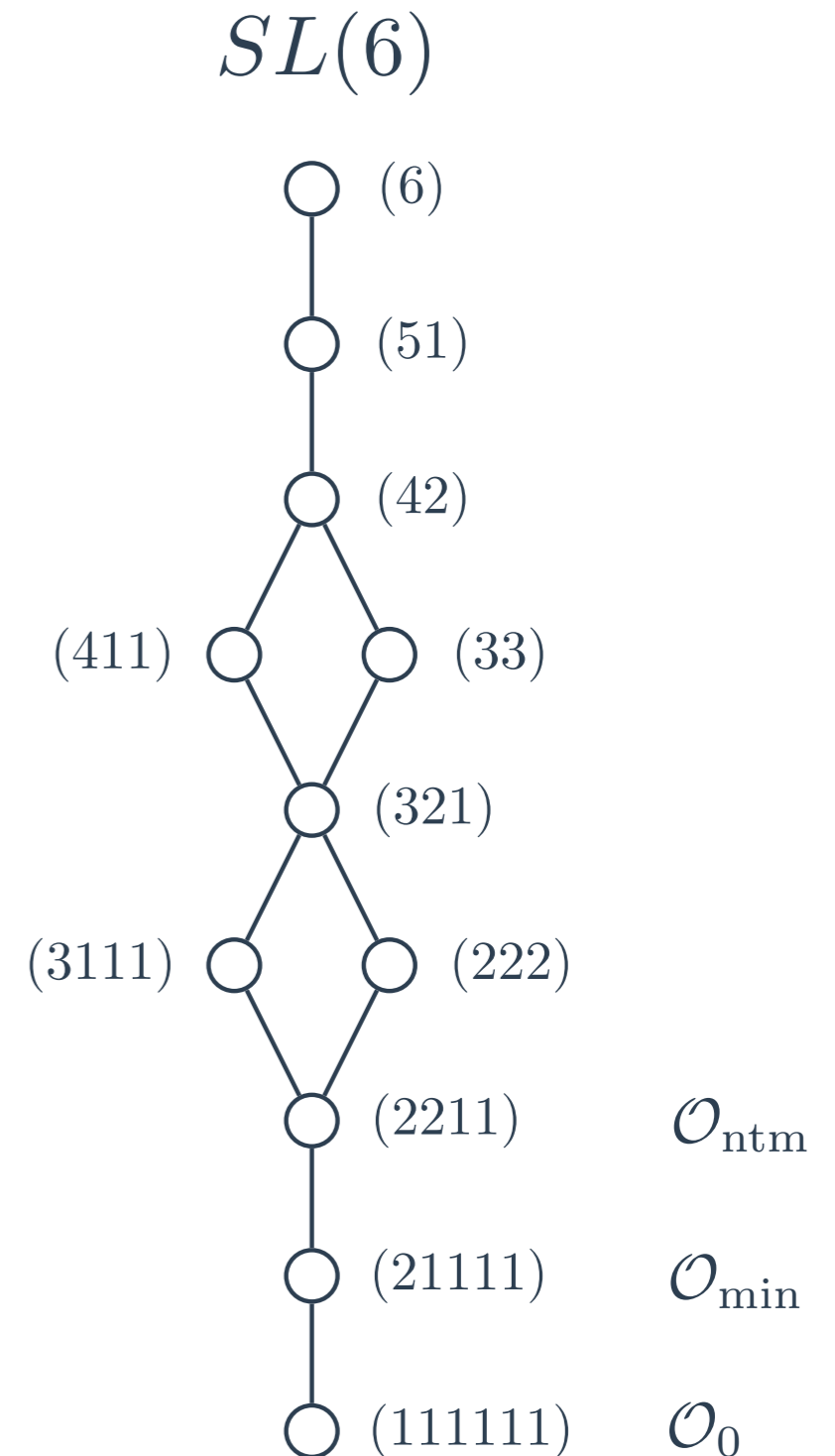
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Closure: $\overline{\mathcal{O}} = \bigcup_{\mathcal{O}' \leq \mathcal{O}} \mathcal{O}'$



Automorphic representations

Small representations

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$$\mathrm{WF}(\pi_{\min}) = \overline{\mathcal{O}_{\min}} = \mathcal{O}_{\min} \cup \mathcal{O}_0$$

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$$\mathcal{E}_0^{(D)} \in \pi_{\min}$$

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[Green-Miller-Vanhove,
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$$\int K K \longrightarrow 0$$

$$\sum K \longrightarrow K$$

Goal: find expressions for Fourier coefficients
in terms of (known) Whittaker coefficients
using vanishing properties of the given π

Previous results

[Miller-Sahi]

Previous results

Theorem

For $G = E_6, E_7$, an automorphic form $\varphi \in \pi_{\min}$ is completely determined by maximally degenerate Whittaker coefficients

W_N with only one $m_\alpha \neq 0$

[Miller-Sahi]

Main results

$SL(3), SL(4)$

[GKP14]

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Corollary

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at most two commuting roots

[GKP14]

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.....
strongly orthogonal

[GKP14]

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Fourier coefficients on maximal parabolic subgroups in the minimal representation



π_{\min}

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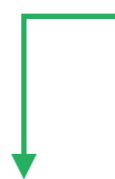
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Maximal parabolic
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Maximal parabolic
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Maximally degenerate

[GKP14]

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○—●

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$$\int_{U(\mathbb{Q}) \backslash U(\mathbb{A})} E(\chi_{\min}; ug) \overline{\psi_U(u)} du = \int_{N(\mathbb{Q}) \backslash N(\mathbb{A})} E(\chi_{\min}; nlg) \overline{\psi'_N(n)} dn$$

Other groups

[Work in progress with Ahlén, Liu, Kleinschmidt, Persson]

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↑
Maximally degenerate

and similar statement for next-to-minimal representation

[Work in progress with Ahlén, Liu, Kleinschmidt, Persson]

Other groups

Conjecture

A similar relations holds for all simple Lie groups

$$F_U(\chi_{\min}, \psi; g) = W_N(\chi_{\min}, \psi'; lg) \quad \text{with } l \in L(\mathbb{Q}) \text{ depending on } \psi$$

↑
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[GKP14]

[Proof in progress with Gourevitch, Kleinschmidt, Persson, Sahi]

Local spherical vectors

Checks for E_6, E_7, E_8

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$$f_{\psi_{U,p}}^\circ \in \text{Ind}_{U(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \psi_{U,p} \quad \text{computed in several cases} \quad p \leq \infty$$

[Dvorsky-Sahi, Kazhdan-Polishchuk, Kazhdan-Pioline, Savin-Woodbury]

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$$W_{\psi'}(\chi_{\min}, 1) = \frac{2}{\xi(4)} \left(\prod_{p < \infty} \frac{1 - p^3 |m|_p^{-3}}{1 - p^3} \right) \left(|m|^{-3/2} K_{3/2}(m) \right)$$

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$$f_{\psi_{U,p}}^{\circ} = \frac{1 - p^3 |m|_p^{-3}}{1 - p^3}$$

[Savin-Woodbury]

$$f_{\psi_{U,\infty}}^{\circ} = m^{-3/2} K_{3/2}(m)$$

[Dvorsky-Sahi]

$$W_{\psi'}(\chi_{\min}, 1) = \frac{2}{\xi(4)} \left(\prod_{p < \infty} \frac{1 - p^3 |m|_p^{-3}}{1 - p^3} \right) \left(|m|^{-3/2} K_{3/2}(m) \right)$$

Local spherical vectors

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The Fourier coefficients capture information about BPS-states and black holes.

Whittaker pairs

Tools for proving the conjecture

[Work in progress with Gourevitch, Kleinschmidt, Persson, Sahi]

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$$(S, f) \in \mathfrak{g} \times \mathfrak{g}$$

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Methods for relating different Whittaker pairs

$$(S, \psi) \longrightarrow (S', \psi')$$

[Work in progress with Gourevitch, Kleinschmidt, Persson, Sahj]

Outlook



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Simplification of Fourier coefficients with χ_{\min} for dimensions lower than three. Kac-Moody groups E_9, E_{10}, E_{11}

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How to define “small automorphic representations” for Kac-Moody groups? What is the mechanism behind the vanishing properties?



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$\mathcal{E}_6 D^6 R^4$ requires extended notion of automorphic forms, the development of which will positively bring new exciting insights to both physics and mathematics.



Thank you!

Henrik Gustafsson

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